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THE PROBLEM OF WAVE MOMENTUM, RADIATION PRESSURE AND OTHER QUANTITIES IN THE CASE OF PLANE MOTIONS OF AN IDEAL GAS†

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Small vibrations and waves of an ideal fluid gas (a liquid or gas) are considered in the quadratic approximation. Actual values of the momentum density and the pressure and also their averaged values are obtained in a number of specific characteristic problems. It is shown that the momentum of an isolated wave with a mean density equal to the density of the unperturbed medium, and the radiation pressure are due to non-linearity of the system of equations, and that this wave has no momentum if its profile remains unchanged during its motion. The latter assertion is also true for finite-amplitude waves. © 1999 Elsevier Science Ltd. All rights reserved.

The action of waves in liquids or gases when reflected from an obstacle, and also the pressure on the wall of a closed volume when these media vibrate is usually explained by the change in the momentum of the perturbations when interaction occurs with the obstacles. The literature contains assertions both regarding the necessary presence of momentum in the waves [1–3], and that there is no such a momentum [4, 5]. Below, in relation to this problem, we investigate the solutions of the equations of hydrodynamics, obtained up to second-order infinitesimals in the amplitude of the perturbations of the parameters of the medium in specific initial-boundary-value problems. Some well-known problem are solved here in a somewhat changed formulation and the results are presented to complete the picture.

1. FORMULATION OF THE PROBLEM

Consider the one-dimensional motion of an ideal fluid (a liquid or gas) in a tube with unit area of cross-section when there are no external forces. The equation of continuity and Euler's equation have the form

$$\rho_t + (\rho v)_x = 0, \quad \rho v_t + \rho v v_x + p_x = 0 \tag{1.1}$$

where υ is the flow velocity at a given point of the medium, ρ is the density and p is the pressure. Multiplying the first equation by υ and adding it to the second, we obtain

$$\rho_t + j_x = 0, \quad j_t + (j^2\rho^{-1} + p)_x = 0 \tag{1.2}$$

where $j = \rho\upsilon$ is the momentum density (the momentum). These equations are simpler than the initial system and more convenient, as will be seen below.

We will consider a wave packet in an unbounded medium, its interaction with a reflecting wall, the generation and development of wave motions of a gas when a piston vibrates, and other problems. Since we are interested in a solution up to second-order infinitesimals in the amplitude of the perturbations of the parameters of the medium, we will seek ρ and j in the form

$$\rho = \rho_0 + \rho' + \rho'' + \ldots, \quad j = j' + j'' + \ldots$$

where the number of primes corresponds to the order of the infinitesimal [6, 7]. Assuming the pressure to be a single-valued function of the density, we will write Poisson's equation

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^\gamma \approx p_0 + c_0^2\rho' + \frac{1}{2}c_0^2(\gamma - 1)\frac{\rho'^2}{\rho_0} + c_0^2\rho'' \tag{1.3}$$

where γ is the adiabatic exponent and $c_0^2 = (\partial p/\partial\rho)_{\rho=\rho_0}$ is the square of the velocity of sound.

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We will write the equations of the first approximation and the obvious solution

$$\rho'_t + j'_x = 0, \quad j'_t + c_0^2\rho'_x = 0$$
$$\rho'/\rho_0 = f_1(\xi = x + c_0t) + f_2(\eta = x - c_0t) \tag{1.4}$$
$$j'/\rho_0 = -c_0f_1(\xi) + c_0f_2(\eta)$$

Here f_1 and f_2 are arbitrary functions characterizing waves travelling to the left and the right, respectively, with constant velocity c_0 without any change in shape, the modulus of these functions being much less than unity.

We will also consider the case when $\gamma < 0$, which is physically inadmissible, since it leads to a pressure drop when the density increases. In this case we can assume (for $\gamma = -1$) [1]

$$p = p_0 + c_0^2\rho_0(1 - \rho_0\rho^{-1}) \tag{1.5}$$

The results obtained below will then be true for any sign of γ.

2. A TRAVELLING WAVE IN A MEDIUM WITHOUT BOUNDARIES

The momentum of a wave packet travelling to the left and described by the continuous function $f(x + c_0t)$, specified in the range $0 \leqslant x + c_0t \leqslant l$ and equal to zero at the ends of this range and outside it, is defined as a whole in accordance with solution (1.4) by a deviation of the density from the equilibrium value. In particular, if at the instant of time t_1 the wave packet lies between x_1 and $x_2 = x_1 + l$ and the excess mass of the packet is positive, i.e.

$$\rho_0\int_{x_1}^{x_2} f(x,t_1)dx > 0$$

then its momentum is

$$\int_{x_1}^{x_2} j'(x,t_1)dx < 0$$

i.e. is in the direction of the wave motion. If there is a mass deficiency, the momentum will be greater than zero, i.e. it will be in the opposite direction to the wave motion. similar conclusions also easily follow for a wave travelling to the right. Thus, in the first approximation the motion of a wave packet which possesses momentum is accompanied by the mass transfer in a direction that is not necessarily the same as its direction of propagation.

The equations of the second approximation for a wave packet travelling to the left have the form

$$\rho''_t + j''_x = 0$$
$$j''_t + c_0^2\rho''_x = -\rho_0^{-1}(j'^2 + \frac{1}{2}(\gamma - 1)c_0^2\rho'^2)_x = -\varepsilon c_0^2\rho_0(f^2(\xi))_x \tag{2.1}$$
$$\varepsilon = (\gamma + 1)/2$$

We will give the solution corresponding to the initial conditions for the second approximation, which is only non-zero when $0 < x < l$

$$\rho''(x,0) = \rho_2^0(x), \quad j''(x,0) = j_2^0(x)$$
$$\rho''(x,t) = \varphi_+(x,t) + \frac{1}{2}[\rho_2^0(\xi) + \rho_2^0(\eta) - c_0^{-1}j_2^0(\xi) + c_0^{-1}j_2^0(\eta)]$$

$$j''(x,t) = \varphi_-(x,t) + \frac{1}{2}[j_2^0(\xi) + j_2^0(\eta) - c_0\rho_2^0(\xi) + c_0\rho_2^0(\eta)] \tag{2.2}$$
$$\varphi_\pm = \frac{\varepsilon}{4}\rho_0[-f^2(\xi) \pm 2c_0t[f^2(\xi)]_x + f^2(\eta)]$$

The solutions are obtained for an arbitrary continuous function $f(x, t)$, which possesses continuous derivatives with respect to x and t, by using the characteristics $\xi = x + c_0t$, $\eta = x - c_0t$ and by converting Eqs (2.1) to these variables, and $t < t_*$ is the time the discontinuity forms.

It follows from the solutions that if at the initial instant, when $\rho_2^0(x) = 0$, $j_2^0(x) = 0$, the wave packet, which is moving to the left, is represented by the function $f(x)$, then, as time passes it disintegrates into a group of waves travelling to the left and with a changing profile (the first two terms), and a wave travelling to the right (the third term). If the excess mass and the momentum of the initial packet, defined by the integral

$$\int_0^l f(x)dx$$

are zero, the two groups of waves which are formed each have separately an excess (or a deficiency) of mass and momentum, defined by the integral of $f^2(x \pm c_0t)$, with these quantities being zero for the whole wave process. This is a manifestation of the laws of conservation of mass and momentum of the whole wave process.

Note that, in contrast, a travelling wave does not arise if

$$\frac{\varepsilon}{2}c_0\rho_0 f^2(x) + c_0\rho_2^0(x) + j_2^0(x) = 0 \tag{2.3}$$

Taking this into account and eliminating $j_2^0(x)$, we will have

$$\begin{aligned} \rho &= \rho_0 + \rho_0 f(\xi) + \frac{\varepsilon}{2}\rho_0 c_0 t[f^2(\xi)]_x + \rho_2^0(\xi) \\ j &= c_0(\rho_0 - \rho) - \frac{\varepsilon}{2}c_0\rho_0 f^2(\xi) \end{aligned} \tag{2.4}$$

We will give one more form of the formulae when

$$\rho_2^0(x) = -\frac{\varepsilon}{2}\rho_0[xf^2(x)]_x$$

We have

$$\begin{aligned} \rho &= \rho_0 + \rho_0 f(\xi) - \frac{\varepsilon}{2}\rho_0[xf^2(\xi)]_x \\ j &= -c_0\rho_0 f(\xi) + \frac{\varepsilon}{2}c_0\rho_0 x[f^2(\xi)]_x \end{aligned} \tag{2.5}$$

Solutions (2.4) and (2.5) differ in the second approximation, which, in the first case, contains a term $\sim t$, and in the second $\sim x$, but the behaviour of the solutions is similar, since when the wave moves $x \sim c_0t$. It follows from (2.4) and (2.5) that

$$\int_{x_1}^{x_2} (\rho' + \rho'')dx \text{ and } \int_{x_1}^{x_2} jdx$$

cannot vanish simultaneously. These formulae do not give as simple a relation between the momentum and the density as in the first approximation.

The solutions for a wave travelling to the right can be obtained by changing the sign of c_0.

From these formulae we obtain, first, a well-known result: a wave cannot propagate without distortions (due to the term proportional to t or x), apart from the case when $\gamma = -1$. Second, a non-decaying wave has, of necessity, either momentum, or an excess (or a deficiency) of mass, or both. Thus, when $\rho_2^0(x) = \varepsilon\rho_0 f^2(x)/2$ the wave packet does not have momentum when there is a mass deficiency.

We will now consider another not unimportant fact which follows from (2.4). When

$$\rho_2^0(x) = 0, \quad \int_0^l f(x)dx = 0$$

the packet possesses zero excess mass, since

$$\int_0^l [f^2(x)]_x dx = 0$$

by virtue of the fact that $f(0) = f(1) = 0$ and, in addition, has a non-zero momentum, equal to

$$\int_0^l j''(x)dx = -\frac{\varepsilon}{2}c_0\rho_0\int_0^l f^2(x)dx \tag{2.6}$$

The following question arises: how can a wave packet without excess mass transport it, since there is a non-zero momentum? We will solve this paradox. We will calculate the momentum of the medium within the limits x_1, x_2 in which the wave packet moves from right to left with a leading wave front at the point $x = -c_0(t)$ and with a rear wave front at the point $x = -c_0(t)$ so that $x_1 \leqslant -c_0t, l - c_0t \leqslant x_2$. The centre of mass x_c of this part of the medium is obtained from the formula

$$\rho_0(x_2 - x_1)x_c = \int_{x_1}^{x_2} x\rho(x,t)dx = \frac{1}{2}\rho_0(x_2^2 - x_1^2) + \rho_0\int_0^l zf^2(z)dz - \frac{\varepsilon}{2}c_0\rho_0 t\int_0^l f^2(z)dz$$

(we have specified the density from (2.4) with $\rho_2^0 = 0$).

Hence it follows that the presence of a wave with zero mean leads, in the first approximation, merely to a constant displacement of the centre of mass of the part of the medium considered, while the velocity of the centre of mass is given by the second approximation as a whole and is due to the change in the wave profile. We have

$$\rho_0 l\dot{x}_c = -\frac{\varepsilon}{2}c_0\rho_0\int_0^l f^2(z)dz$$

which is identical with the expression for the momentum (2.6). The result obtained can also be extended to a wave of finite amplitude.

Hence, despite the fact that there is no excess mass, the momentum of the packet is also related to the transfer of mass due to redistribution of the density relative to the packet, i.e. due to a change in its profile. The momentum is proportional to $\gamma + 1$, the denser parts possess greater velocity than the less dense parts [8], and hence its direction coincides with the direction of the packet propagation if $\gamma > -1$. The fact that the directions of the momentum and the propagation velocity of the wave packet are not the same when there is a mass deficiency and particularly when momentum is present in the case of zero excess mass of the packet, changes the relation between the momentum and the velocity considerably in discrete systems and continuous media. Whereas in a discrete system, negative mass can serve as the analogue of the first fact, there is no analogue of the second fact, it is inherent only in a distributed system and is due to its non-linearity, and the sign of the effect is determined by the sign of $\gamma + 1$. Note that when $\gamma = -1$ the equations in Lagrangian coordinates are linear [7] and the waves propagate without distortion, exactly as in Euler coordinates, and hence the second fact mentioned above is not present in this case and a travelling wave with zero excess mass has no momentum [9].

It should be noted that to determine the momentum (2.6) the first approximation is insufficient, unlike the case of a wave occupying a finite region of three-dimensional space, and one also cannot use the constancy of the velocity potential outside the wave [8]. In the case of plane waves the velocity potentials in front of and behind the wave are different.

3. INTERACTION OF A WAVE PACKET OF ARBITRARY FORM AND REFLECTED FROM AN IMPENETRABLE WALL

Suppose $x = 0$ corresponds to the wall and that the wave packet at the initial instant of time is situated in the region $x_0 < x < l + x_0$ and moves towards the wall. When $x_0 \leqslant c_0t \leqslant x_0 + l$ the wave packet interacts with the wall and when $x_0 + l \leqslant c_0t$ the reflected wave packet which is formed moves away. Since the wall is impenetrable we can write the condition $j(0, t) = 0$.

To solve this boundary-value problem we will use a well-known method and consider the unbounded problem, which is symmetrical with respect to $x = 0$, in which

$$j(x,t) = -j(-x,t), \quad \rho(x,t) = \rho(-x,t)$$

while the initial values $j_0(x)$ and $\rho_0(x)$ are taken to be such that non-decaying wave packets are excited, situated symmetrically with respect to the wall and travelling towards it. We will write this solution of this problem in the form

$$\rho=\rho_0\left\{1+f(\xi)+\frac{\varepsilon}{2}[f^2(\xi)]_x c_0 t+\rho_0^{-1}\rho_2^0(\xi)+f(\eta)-\frac{\varepsilon}{2}[f^2(\eta)]_x c_0 t+\right.$$
$$\left.+\ \rho_0^{-1}\rho_2^0(\eta)+\frac{1}{4}(3-\gamma)[f_x(\eta)F(\xi)+f_x(\xi)F(\eta)+2f(\xi)f(\eta)]\right\} \qquad (3.1)$$
$$j=+\rho_0 c_0\left\{-f(\xi)-\frac{\varepsilon}{2}f^2(\xi)-\frac{\varepsilon}{2}[f^2(\xi)]_x c_0 t-\rho_0^{-1}\rho_2^0(\xi)+f(\eta)+\right.$$
$$\left.+\frac{\varepsilon}{2}f^2(\eta)-\frac{\varepsilon}{2}[f^2(\eta)]_x c_0 t+\rho_0^{-1}\rho_2^0(\eta)+\frac{1}{4}(3-\gamma)[f_x(\eta)F(\xi)-f_x(\xi)F(\eta)]\right\}$$
$$x_0\le\xi\le l+x_0,\quad -l-x_0\le\eta\le -x_0$$

where $F_z(z) = f(z)$, $\rho_2^0(x, 0)$ are the initial data for the second approximation and $j(0, t) = 0$ by virtue of the fact that the functions $f(x, t)$ and $\rho_2^0(x, 0)$ are even with respect to $x = 0$.

The solutions for $j(x, t)$ and $\rho(x, t)$ can be treated in two ways: (1) as a description of the wave process of approaching wave packets when $0 \leqslant c_0 t \leqslant x_0$, when their domains do not overlap and hence the last terms containing the products of functions of ξ and η are zero, the interaction of wave packets when $x_0 \leqslant c_0 t \leqslant l + x_0$, when all terms of the solutions are non-zero, and finally, the diverging of wave packets when $x_{0+l} \leqslant c_0 t$ and the last terms of the solutions again vanish; (2) as a description of wave packets approaching the wall, when the wave packets are situated as a whole to the right of it when $0 \leqslant c_0 t \leqslant x_0$, when only terms which depend on ξ need be taken into account, and the interaction of a wave packet with the wall when $x_0 \leqslant c_0 t \leqslant x_0 + l$ (all terms with $0 < x$ are taken into account) and which moves away from the wall after a packet is reflected when $l + x_0 \leqslant c_0 t$, when only terms which depend on η need be taken into account.

Substituting $\rho(0, t)$ into (1.3), we obtain a value of the wave pressure on the wall at any instant of time and the overall interaction

$$\int_{x_0/c_0}^{(x_0+l)/c_0} p(0,t)dt=[2\rho_0\overline{f(c_0 t)}+\varepsilon\rho_0\overline{f^2(c_0 t)}+2\overline{\rho_2^0(c_0 t)}]c_0 l$$

We will now calculate the initial momentum of the wave packet when $x > 0$

$$G(x,0)=\int_{x_0}^{l+x_0} j(x,0)dx=-\left[\rho_0 c_0\overline{f(x)}+\frac{\varepsilon}{2}\rho_0 c_0\overline{f^2(x)}+c_0\overline{\rho_2^0(x)}\right]l$$

The momentum of the reflected packet $G(x, t_1)$ will have the same value but opposite sign for $x_0 + l < c_0 t_1$ and $0 < x$, and $G(x, t_1) - G(x, 0)$ is identical with the overall action of the wave packet on the wall. In the case of a wave without a density excess its momentum and interaction with the wall depend on the length of the wave packet, the form of the initial perturbation $f(x, 0)$ and the non-linear properties of the medium. When $1 + \gamma > 0$ the wave exerts pressure on the wall but when $1 + \gamma < 0$ the pressure on the wall is negative.

4. THE EXCITATION OF WAVES IN A GAS BY A MOVING PISTON

We will consider the motion of a gas both from the right and from the left from a piston which is situated at the origin of the system of coordinates when $t = 0$, where the piston velocity, like the gas velocity, is everywhere zero. Consequently $j(x, 0) = 0$ for all x.

We will derive the boundary conditions by considering the problem for small vibration of the piston. The gas velocity v at the point $s(t)$ where the piston is situated must be equal to its velocity

$$v(s(t),t)=\dot{s}(t)\approx v'(0,t)+v''(0,t)+v'_x(0,t)s(t)$$

Taking the relation $j(x, t) = (\rho_0 + \rho')(v' + v'')$ into account we obtain

$$j'(0,t)=\rho_0\dot{s}(t);\quad j''(0,t)=-j'_x(0,t)s(t)+\rho'(0,t)\dot{s}(t)$$

The equations of the first approximation, and the boundary and initial conditions are

$$\rho'_t + j'_x = 0, \quad j'_t + c_0^2\rho'_x = 0, \quad j'(0,t) = \rho_0\dot{s}(t), \quad 0 \le t < \infty$$
$$\dot{s}(0) = 0, \quad j'(x,0) = 0, \quad \rho'(x,0) = 0, \quad -\infty < x < +\infty$$

For the second approximation we have

$$\rho''_t + j''_x = 0, \quad j''_t + c_0^2\rho''_x = -\tfrac{1}{2}[c_0^2\rho_0^{-1}(\gamma-1)\rho'^2 + j'^2\rho_0^{-1}]_x$$
$$j''(0,t) = \rho'(0,t)\dot{s}(t) - j'_x(0,t)s(t), \quad j''(x,0) = 0$$

The solution of the problem of the first approximation has the form

$$c_0\rho' = \Theta(x)\rho_0 s_t(z_-) - \Theta(-x)\rho_0 s_t(z_+)$$
$$j' = \Theta(x)\rho_0 s_t(z_-) + \Theta(-x)\rho_0 s_t(z_+)$$
$$z_\pm = t \pm x/c_0$$

where $\Theta(x)$ is the unit function and $\Theta(0) = 1/2$.

The solution consists of two waves of the same kind starting from the point $x = 0$, propagating in opposite directions with the leading wave front of the wave travelling to the right (to the left) at the point $x = c_0t(-c_0t)$.

The equations of the second approximation for the wave travelling to the right have the form

$$\rho''_t + j''_x = 0, \quad j''_t + c_0^2\rho''_x = -\varepsilon\rho_0[s_t^2(z_-)]_x, \quad 0 \le x \le c_0t$$

For the wave travelling to the left, where $-c_0t \leqslant x \leqslant 0$, we must replace c_0 by $-c_0$.

The initial conditions are $j''(x, 0) = 0$, $\rho''(x, 0) = 0$. The boundary conditions, taking the solution of the first approximation into account, can be reduced to the form

$$j''(0,t) = \rho_0 c_0^{-1}[s(t)s_t(t)]_t = R_t(t)$$

The solution of the problem for the wave travelling to the right is

$$\rho' + \rho'' = \rho_0 c_0^{-1} s_t(z_-) - \frac{\varepsilon}{2}\rho_0 c_0^{-2}[x s_t^2(z_-)]_x + c_0^{-1}R_t(z_-)$$

$$j' + j'' = \rho_0 s_t(z_-) - \frac{\varepsilon}{2}\rho_0 c_0^{-1} x[s_t^2(z_-)]_x + R_t(z_-)$$

For the wave travelling to the left we must change the sign of c_0.

To the expressions for ρ', j' we must add the solutions in the form of waves with amplitude which increases as x increases, proportional to $\gamma + 1$, and waves determined by the piston motion in terms of the function $R_t(t)$. Terms proportional to x lead to distortion of the wave profile, and it is these, together with first-order terms, that are responsible for the wave momentum. If the piston motion in terms of the function $R_t(t)$. Terms proportional to x lead to distortion of the wave profile, and it is these, together with first-order terms, that are responsible for the wave momentum. If the piston motion ceases at a certain time $t = T$, two wave packets are formed at this instant, which travel in opposite directions. When $t > T$ their leading wave fronts are at distances of $x = \pm c_0t$, while the rear wave fronts are at distances of $x = \pm c_0(t - T)$.

We will calculate the momentum and mass excess of these packets for arbitrary motion $s(t)$ of the piston under the following conditions

$$s(0) = 0, \quad s(t \ge T) = s(T), \quad s_t(0) = s_t(T) = 0$$

We obtain

$$G_\pm = \int_{x_1}^{x_2} (j' + j'')dx = \rho_0 c_0 s(T) \pm \frac{\varepsilon}{2}\rho_0 \int_0^T s_t^2(\tau)d\tau$$

$$\int^{x_2} (\rho' + \rho'')dx = \pm\rho_0 s(T)$$

where $x_1 = +c_0(t - T)$, $x_2 = c_0t$ for $x > 0$ and $x_1 = -c_0t$, $x_2 = -c_0(t - T)$ for $x < 0$. The upper (lower) sign corresponds to a packet which is situated at $x > 0$ ($x < 0$) and travels to the right (to the left). The momentum, in the first approximation, is determined by the resultant displacement $s(T)$ of the piston. Here the gaseous medium, both on the right and on the left, obtains the same momentum in the direction in which the piston is displaced, despite the fact that the wave packets move in opposite directions. The density excess is also determined by the displacement $s(T)$ of the piston, and has opposite signs for waves travelling in opposite directions. When $s(T) > 0$ a compression of the medium occurs on the right of the piston and a rarefaction on the left of it.

The second term in the formulae for the momentum is due to terms of second-order infinitesimals. However, when $s(T) = 0$, i.e. when the piston ceases to move at the initial point, or when the time T is long, this part of the momentum becomes predominant. It is always directed in the direction of propagation of the wave packet if $\gamma + 1 > 0$ and in the opposite direction when $\gamma + 1 < 0$.

We will obtain the conditions for which the second term in the formula for the momentum is much greater than the first for a piston which vibrates as given by the following relation

$$s(t) = \begin{cases} a(1-\cos\omega t), & 0 \le t \le n\pi/\omega = T \\ s(t) = s(T), & T < t \end{cases}$$

Here the total time of motion of the piston is equal to an integer number of half-periods of its vibration. The momenta of the wave packets for odd n are

$$G_{\pm} = 2\rho_0 c_0 a \pm \frac{\varepsilon}{4}\rho_0\omega^2 a^2 T$$

and the required conditions can be reduced to the form $n \gg 8c_0/(\pi\varepsilon a\omega)$.

An estimate shows that, even for very high-power radiators, this relation has a large value and the first term can only be neglected for long wave trains When $s(T) = 0$ wave packets are generated without excess mass, but with a non-zero momentum. This is in complete agreement with the results obtained in Section 2 and is due to the change in the wave profile. The mass flow through an arbitrary cross-section x during the time between the arrival of wave packets t_1 and their completion t_2 is zero.

The results of the solution of this problem are comparable with Earnshaw's solution [7].

5. TWO FIXED WALLS

Consider the vibrations of a gas with the following boundary and initial conditions

$$j(0,t) = j(l,t) = 0, \quad \rho(0,x) = \rho_0 + \rho_0 ak\cos kx, \quad j(0,x) = 0$$

where $\sin kl = 0$, $ak \ll 1$, $c_0 = \omega/k$. We have

$$\rho = \rho_0 + \rho_0 ak\cos kx\cos\omega t + \frac{1}{4}\rho_0 a^2k^2\cos 2kx[(2-\varepsilon)(1-\cos 2\omega t) - \varepsilon\omega t\sin 2\omega t]$$

$$j = \rho_0 a\omega\sin kx\sin\omega t + \frac{1}{4}\rho_0 a^2 k\omega\sin 2kx\left[\left(\frac{3}{2}\varepsilon - 2\right)\sin 2\omega t + \varepsilon\omega t\cos 2\omega t\right]$$

Here the average radiation pressure is

$$\overline{\Pi}' = \overline{p + \rho v^2}' = \frac{1}{\tau}\int_{t_0}^{t_0+\tau}\Pi(x,t)dt = p_0 + \frac{\varepsilon}{4}\rho_0 a^2\omega^2\left(1 + \frac{1}{2}\cos 2\omega t_0\cos 2kx\right)$$

Here τ is the averaging time, which is equal to an integer number of periods of the gas vibrations.

As can be seen, the average radiation pressure has a constant component and a component that varies in space [7]. However, the latter depends on the origin t_0 of the averaging process and is determined as a whole by the secular term of the solution $\rho(x, t)$. This means that the dependence of the average pressure on x is due to the unsteady nature of the vibrations. Taking into account that t_0 is arbitrary, we can drop the last term in the expression for the average of Π. We then obtain an expression for the radiation pressure which agrees with that obtained previously [9] without using the exact solution.

6. TWO VIBRATING WALLS PRODUCING A TRAVELLING WAVE

Suppose two impenetrable solid walls vibrate in synchronism, maintaining a constant distance l between them and producing a travelling wave.

When $lk = 2\pi n$ the solutions have the form

$$\rho = \rho_0 + \rho_0 ak \sin(\omega t - kx) + \frac{\varepsilon}{4}\rho_0 a^2 k^2 [2\omega t \sin 2(\omega t - kx) + \cos 2(\omega t - kx)]$$

$$j = \rho_0 a\omega \sin(\omega t - kx) + \frac{\varepsilon}{2}\rho_0 a^2 \omega^2 kt \sin 2(\omega t - kx)$$

The vibration of the walls and the initial conditions have the form

$$s_1(t) = s_2(t) = -a\cos\omega t + \frac{1}{8}a^2 k[(4+\varepsilon)\sin 2\omega t - 2\varepsilon\omega t \cos 2\omega t]$$

$$\rho(x,0) = \rho_0 - \rho_0 ak \sin kx + \frac{\varepsilon}{4}\rho_0 a^2 k^2 \cos 2kx, \quad j(x,0) = -\rho_0 a\omega \sin kx$$

The solutions satisfy the relations

$$\int_{s_1}^{l+s_1} \rho(x,t)dx = \rho_0 l, \quad \int_{s_1}^{l+s_1} j(x,t)dx = 0$$

which denote that the mass of gas between the walls remains unchanged and that this gas has no momentum at any instant of time.

When deriving the solutions we used the relations

$$v(l+s_2,\ t) = \dot{s}_2, \quad j = \rho v$$

and the expressions for the approximations of s_2 were obtained from the formulae

$$\rho_0 \dot{s}_2' = j'(l,t), \quad \rho_0 \dot{s}_2'' = (j_x' s_2' + j'' - \rho_0^{-1}\rho' j')_{x=l}$$

Similar expressions for s_1 are obtained from these by making the replacement $1 \to 0$. This applies also to the problems considered below in Sections 7 and 8. The average value of the radiation pressure will be

$$\overline{\Pi}' = \frac{1}{\tau}\int_{t_0}^{\tau+t_0} \Pi dt = p_0 + \frac{\varepsilon}{2}\rho_0 a^2 \omega^2 \left[1 - \frac{1}{2}\cos 2(\omega t_0 - kx)\right]$$

As in the previous problem, the result of averaging depends on t_0. The additional pressure that arises due to excitation by a travelling wave is double that obtained in the case of a standing wave. This is explained by the fact that for a travelling wave the amplitude of the gas vibrations in each cross-section x is the same, whereas for a standing wave this amplitude is only reached an antinodes. In this case (Sections 5 and 6) the radiation pressure is not related to the gas momentum, which is always zero.

7. VIBRATIONS OF A GAS BETWEEN TWO WALLS— FIXED AND SPRING-LOADED

The boundary conditions on the fixed and moving walls have the form

$$j(0,\ t) = 0, \quad p(l+s_2,\ t) = rs_2 + p_0$$

where s_2 is the displacement of the right-hand wall and r is the stiffness of the spring.

Here steady vibrations are possible in the form of standing waves

$$\rho = \rho_0 + \rho_0 ak \cos kx \cos\omega t +$$
$$+ \frac{1}{4}\rho_0 a^2 k^2 \{(2\cos 2kx + \varepsilon kx \sin 2kx)\cos 2\omega t + (2-\varepsilon)\cos 2kx\} + \rho_0''$$

$$j = \rho_0 a\omega \sin kx \sin \omega t + \frac{1}{4}\rho_0 a^2 k\omega\left[\left(2+\frac{\varepsilon}{2}\right)\sin 2kx - \varepsilon kx \cos 2kx\right]\sin 2\omega t$$

$$s_2 = -a \sin kl \cos \omega t + \frac{\varepsilon}{8}a^2 k\left[-\frac{\sin 2kl}{2} + kl \cos 2kl\right]\cos 2\omega t + s_{20}$$

where

$$\rho_0'' = -\frac{\varepsilon}{4}\rho_0 a^2 k^2 \frac{\rho_0 c_0^2}{\rho_0 c_0^2 + rl}\left(1 + \frac{r^2}{\rho_0^2 c_0^2 \omega^2 + r^2}\right)$$

$$\overline{\Pi}' = p_0 + \frac{\varepsilon}{4}\rho_0 a^2 \omega^2\left(1 + \frac{\sin 2kl}{2kl}\right)\frac{rl}{rl + \rho_0 c_0^2}$$

$$s_{20} = \frac{\varepsilon}{4}a^2 k^2 \frac{\rho_0 c_0^2 l}{\rho_0 c_0^2 + rl}\left(1 + \frac{\sin 2kl}{2kl}\right)$$

The parameters and the solutions satisfy the relations

$$\operatorname{ctg} kl = -r(\rho_0 c_0 \omega)^{-1}, \quad kl \operatorname{ctg}^2 kl + 3 \operatorname{ctg} kl + 3kl = 0$$

$$\int_0^{l+s_2} \rho(x, t)dx = \rho_0 l$$

The average values of the parameters for the limit values of the spring stiffness are of interest, namely,

$$\Pi_0'' \to 0, \quad \rho_0'' \to -\frac{\varepsilon}{4}\rho_0 a^2 k^2, \quad s_{20} \to \frac{\varepsilon}{4}a^2 k^2 l \quad \text{as} \quad r \to 0$$

$$\Pi_0'' \to \frac{\varepsilon}{4}\rho_0 a^2 \omega^2, \quad \rho_0'' \to 0, \quad s_{20} \to 0 \quad \text{as} \quad r \to \infty$$

i.e. the radiation pressure in the case of a very high stiffness approaches the pressure which occurs when the gas vibrates in the form of standing waves with fixed walls, whereas for a very soft spring it approaches zero with a maximum increase in the volume due to s_{20} and a reduction in the density.

8. A STEADY TRAVELLING WAVE

A steady travelling wave occurs for certain vibrations of the left-hand wall and when the right-hand wall, which has mass, has an elastically damped fixing. The boundary condition at the right end has the form

$$p(l + s_2, t) = m\ddot{s}_2 + h\dot{s}_2 + rs_2 + p_0$$

A travelling wave occurs when $m\omega^2 = r$, $h = \rho_0 c_0$ and can be written in the form

$$\rho = \rho_0 + \rho_0 ak \sin(\omega t - kx) + \frac{1}{2}\rho_0 a^2 k^2\left\{\frac{\varepsilon - 4}{2}\cos 2(\omega t - kx) + \right.$$

$$\left. + \left[\varepsilon k(x - l) - \frac{\varepsilon}{3r}\rho_0 c_0 \omega\right]\sin 2(\omega t - kx) - \varepsilon\frac{\rho_0 c_0^2}{\rho_0 c_0^2 + rl}\right\}$$

$$j = \rho_0 a\omega \sin(\omega t - kx) + \rho_0 a^2 k\omega\left\{-\cos 2(\omega t - kx) + \left[\frac{\varepsilon k}{2}(x - l) - \frac{\varepsilon}{6r}\rho_0 c_0 \omega\right]\sin 2(\omega t - kx)\right\}$$

The left-hand wall vibrates with zero mean as given by the relation

$$s_1 = -a \cos \omega t + \frac{1}{4}a^2 k\left[\varepsilon kl + \frac{\rho_0 c_0 \omega}{3r}\varepsilon\right]\cos 2\omega t$$

while the right-hand wall vibrates as given by the relation

$$s_2 = -a\cos\omega t + \frac{\varepsilon}{12r}\rho_0 a^2\omega^2\cos 2\omega t + \frac{\varepsilon}{2}a^2k^2l\frac{\rho_0 c_0^2}{\rho_0 c_0^2 + rl}$$

The average radiation pressure in this case is

$$\overline{\Pi(x,\ t)}^t = p_0 + \frac{\varepsilon}{2}\rho_0 a^2\omega^2\frac{rl}{\rho_0 c_0^2 + rl} \quad \text{when} \quad \int_{s_1}^{l+s_2}\rho(x,\ t)dx = \rho_0 l$$

The average pressure is determined not only by the amplitude of the wave and the non-linearity but also by the stiffness r of the spring restraining the right-hand wall. We have

$$\Pi_0'' \to 0,\quad s_{20} \to \frac{\varepsilon}{2}a^2k^2l,\quad \rho_0'' \to -\frac{\varepsilon}{2}\rho_0 a^2k^2 \quad \text{as} \quad r \to 0$$

$$\Pi_0'' \to \frac{\varepsilon}{2}\rho_0 a^2\omega^2,\quad s_{20} \to 0,\quad \rho_0'' \to 0 \quad \text{as} \quad r \to \infty$$

Thus, the average pressure which occurs when there is a travelling wave is a maximum when $r = \infty$ and is equal to the pressure obtained in Section 6.

9. THE SOUND ENERGY

In [7] two expressions were derived for the sound energy density E. In the quadratic approximation

$$E_2 = \frac{\rho_0 v'^2}{2} + \frac{c_0^2\rho'^2}{2\rho_0} + \frac{c_0^2}{\gamma - 1}(\rho' + \rho'')$$

$$E_{2A} = \frac{\rho_0 v'^2}{2} + \frac{c_0^2\rho'^2}{2\rho_0} + \frac{c_0^2}{\gamma}(\rho' + \rho'')$$

The first of these expressions is the generally accepted one, while the second was proposed by Andreyev [6]. Note that when the volume V_0 remains unchanged

$$\int_{V_0}(\rho' + \rho'')dV = 0$$

and hence, to calculate the total sound energy, we can assume the density: as given by the first two similar terms of these expressions. However, in general (in the case of varying surfaces or when calculating the energy in part of the volume) one cannot use a truncated expression for the density. One can use the solutions obtained to solve the problem in favour of one or other definition of E. The difference is considerable and is found even in the first order. This considerably simplifies the problem.

The potential energy density is given by the following expression [7, 8]

$$\rho u = \frac{c_0^2}{\gamma(\gamma - 1)}\frac{\rho^\gamma}{\rho_0^{\gamma-1}} \approx \frac{\rho_0 c_0^2}{\gamma(\gamma - 1)} + \frac{\rho' c_0^2}{\gamma - 1}$$

where u is the potential energy per unit mass and the second term of this expression is usually employed as a linear correction to the density of the unperturbed medium when it vibrates. However, the change in volume is ignored here. In the first approximation we write the energy of the mass of gas in a small volume V_0 before and $V_0 + V'$ after the perturbation so that we can assume the quantity $(\rho u)'$ to be unchanged within this volume. We have

$$\int_{V_0 + V'}[\rho_0 u_0 + (\rho u)']dV \approx \rho_0 u_0(V_0 + V') + (\rho u)'V_0$$

Subtracting the energy of the unperturbed medium and dividing by the volume we obtain the sound energy density in the first approximation as obtained by Andreyev

$$E_{1A} = \rho_0 u_0 \frac{V'}{V_0} + (\rho u)' = \frac{\rho' c_0^2}{\gamma}$$

Here we have used the relation ρV = const, which indicates that the mass of the part of the medium considered remains unchanged.

An expression for the sound energy density was derived in the general case in [6].

As an example consider the problem from Section 4 with the solution

$$\rho' = \rho_0 c_0^{-1} s_t (t - x / c_0)$$

In this problem all the energy of the wave travelling to the right is determined in the first approximation by the work done by the pressure forces p_0 on the wall and its displacement $s(T)$ and is equal to $p_0 s(T) = c^2{}_0 \rho_0 s(T)/\gamma$, where T is the time the piston stops. On the other hand, this energy can be obtained by integrating with respect to x the energy density of the wave formed. Taking the energy density as given by Andreyev, we have

$$\int_0^{c_0 t} \frac{c_0^2}{\gamma} \rho' dx = \frac{\rho_0 c_0}{\gamma} \int_0^{c_0 t} s_t \left(t - \frac{x}{c_0} \right) dx = \frac{\rho_0 c_0^2}{\gamma} s(T) \quad (T < t)$$

Agreement between the results of calculations of the energy is confirmed by the correctness of the expression for the energy density as given by Andreyev.

The case when there is no mass flux ($\bar{j}^t = 0$) is of interest. This occurs in the problem from Section 4 when $\bar{s}_t' = 0$. We will take, for example, the solution for a wave travelling to the right and calculate the time-average density using the standard formula

$$\bar{E}_2 = \rho_0 \overline{s_t^2} \left(t - \frac{x}{c_0} \right) - \frac{\varepsilon \rho_0}{2(\gamma - 1)} \bar{s}_t^2 = \frac{3\gamma - 5}{4(\gamma - 1)} \rho_0 \overline{s_t^2}$$

This result is inapplicable from the physical point of view since for $1 < \gamma < 5/3$, i.e. for the majority of gases, the energy in the sound field is less than the energy in the unperturbed medium, which leads to instability. Attention has already been drawn to this result in [7, 10], but another treatment of the reason for its occurrence is given. If we take the energy density as derived by Andreyev, calculation gives

$$\bar{E}_{2A}^t = \frac{3\gamma - 1}{4\gamma} \overline{s_t^2} > 0$$

for all gases.

10. THE AVERAGE QUANTITIES

We are usually most interested in time-averaged quantities rather than in the instantaneous values of the quantities representing the wave or vibration processes. The time and space averages are often identical but, for example, this is not so in the case waves with varying profile. To answer the question of which averages should be used, we will consider the equation which relates the energy density and its flux q

$$E_t + q_x = 0$$

$$\int_{x_1}^{x_2} [E(x,\ t_2) - E(x,\ t_1)] dx + \int_{t_1}^{t_2} [q(x_2,\ t) - q(x_1,\ t)] dt = 0$$

Consider the case when a wave perturbation propagates from left to right so that its leading wave front when $t = t_1$ is in the section x_1, and when $t = t_2$ it is in the section x_2. Then

$$\int_{x_1}^{x_2} E(x,\ t_2) dx = \int_{t_1}^{t_2} q(x_1,\ t) dt \quad (x_2 - x_1) \bar{E}^x = (t_2 - t_1) \bar{q}^t$$

and, taking into account the equation $x_2 - x_1 = c_0(t_2 - t_1)$, we obtain $c_0 \bar{E}^x = \bar{q}^t$. Hence, the widely used formula $c_0\bar{E}^x = \bar{q}^t$ is unsuitable for waves with varying profile, and to calculate the energy in a specified volume one must use its spatial density or its time-average flux through a specified surface. This also applies to the equations

$$\rho_t + j_x = 0, \quad j_t + \Pi_x = 0$$

where, when calculating the mass and the mass flux (the first equation) one must use the averages $\bar{\rho}^x$, $\bar{j}^t$, and when calculating the momentum and the radiation pressure (the second equation) one must use $\bar{j}^x$, $\bar{\Pi}^t$.

We will give a formula which expresses the radiation pressure in terms of the kinetic energy density for a fixed distance between the walls (Sections 5 and 6) and in the case of spring-loaded walls (Sections 7 and 8)

$$\bar{\Pi}''^t = \varepsilon \bar{\bar{E}}_k$$

In the case of the problems from Sections 7 and 8, the right-hand side of the equation must be supplemented with the factor $rl/(rl + \rho_0 c_0^2)$. Note that the radiation pressure depends very much on the stiffness r of the spring.

11. CONCLUSION

The presence of momentum in an isolated wave is determined by the initial conditions. A wave without an excess or a deficiency of density has momentum in the same direction as its motion when $\gamma + 1 > 0$, and an indication of the presence of momentum in this case is the change in its profile. The mass transfer is a necessary consequence of the passage through the medium of a wave having momentum.

One must distinguish the reasons for the occurrence of an average radiation pressure on a reflecting obstacle in the case of an isolated wave and in the case of a closed volume, filled with a vibrating medium. In the first case the pressure is due to the presence of momentum in the wave, while in the second it is due to the tendency of the medium to expand when vibrations occur (in the form of standing or travelling waves) due to the non-linearity of the equations and the medium. The quantity $\gamma + 1$ is a measure of the non-linearity. The average pressure on the wall of a closed volume depends very much on the forces restraining these walls.

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